Mathematical analysis of parametric resonance, from the Mathieu equation to QASER

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Abstract

A periodic driving coefficient in a system of differential equations which are otherwise "conservative" generally has solutions that are not periodic. Historically, in mathematics and engineering, researchers have emphasized the study of the so-called "stable" solutions, namely, those that decay with time. However, in [1], Svidzinsky, Yuan and Scully noticed that the "unstable" solutions are "more useful" in the sense that the gains or exponential growth of some solutions of the system can help a novel design of light amplifiers acronymed QASER (quantum amplification by superradiant emission of radiation).

In order to understand how parametric resonance works, two basic mathematical models are investigated. The first is the well known Mathieu equation in mathematical physics:

\[ \ddot{y}(t) + \left[ \omega_0^2 - 2q \cos(2t) \right] y(t) = 0, \]

while the second is the system of coupled oscillators (QASER):

\[
\begin{align*}
\ddot{\phi}_1 + \omega_0^2 \phi_1 \cdot \Omega^2 \phi_2 &= 0, \\
\ddot{\phi}_2 + \omega_0^2 \phi_2 \cdot \Omega^2[1 + \delta \cos(2t)] \phi_1 &= 0.
\end{align*}
\]

Note that the time variable \( t \) above has been normalized, and the quantities \( q, \delta \) and \( \Omega \) are small.

The rates of gain as a function of \( \omega_0 \) can be viewed in Figure 1 and 2. One notes by comparing them that The QASER system (2) is able to shift the maximum gain much farther to the right with higher frequencies.

Mathematically, our work here is to study such gains and investigate their parametric resonance spectral patterns. We use the Floquet theory and develop a projection method ([2]) that can properly capture gains near the primary resonance and subharmonic frequencies for the Mathieu equation and QASER.